An Eigenadmittance Condition Applicable to Symmetrical Four-Port Circulators and Hybrids

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Abstract—Recently, an expression for the equivalent admittance of symmetrical three-port circulators has been derived on the basis of symmetry considerations. In this paper, the same technique is extended to the symmetrical four-port circulator. It is shown that a simple relationship must be satisfied by the eigenadmittances jY_0, jY_{-1}, jY_1, and jY_2 if perfect circulation is to be obtainable by matching alone. This relationship also applies to reciprocal four-port devices with 2-fold symmetry about two perpendicular axes which are required to be 90° hybrids upon matching, provided that the eigenadmittances jY_1, jY_2, jY_3, and jY_4 are used. An interesting application of this relation to the construction of a compact stripline hybrid is given.

I. Introduction

In THE CASE OF the three-port circulator with 3-fold symmetry, it is often important to know what identical two-port matching network must be connected in each of the circulator arms in order to match the circulator satisfactorily over some prescribed bandwith. It is known that this matching network must match into the input admittance at port 1, which results when a tuner in port 2 is adjusted to give perfect isolation at port 3 [1]. This fact may be used to derive expressions for this input admittance, which is sometimes referred to as the complex gyrator admittance, in terms of quantities which characterize the junction [2]–[4]. Recently, an alternate derivation based on the symmetry properties of the junction has been given, and the resulting quantity was called the equivalent admittance [5], [6]. One advantage of the latter procedure is that it can be readily generalized to other networks with symmetry and where one can again speak of an equivalent admittance. In this paper, such a procedure is carried through for symmetrical four-port circulators. It is shown that a simple relationship must be satisfied by the circulator eigenadmittances jY_0, jY_1, jY_{-1}, and jY_2 if perfect circulation is to be obtainable by matching alone.

II. The Equivalent Admittance at a Reference Plane with $jY_0 = j\infty$

It is known that the phases ψ_0', ψ_{-1}', ψ_1', and ψ_2' of the four eigenreflection coefficients of a four-port circulator must differ from each other by 90° for perfect circulation. That is

$$\begin{aligned}\psi_1' &= \psi_0' + \pi/2\\ \psi_{-1}' &= \psi_0' - \pi/2\\ \psi_2' &= \psi_0' + \pi\end{aligned} \tag{1}$$

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where primed quantities refer to quantities obtained after identical matching networks have been connected in each port. If Y_i and Y_i' $(i = 0, -1, 1, 2)$ are the eigensusceptances before and after matching, then

$$\begin{aligned}-Y_0' &= \tan\left(\frac{\psi_0'}{2}\right) = -\frac{C+DY_0}{A-BY_0}\\ -Y_{-1}' &= \tan\left(\frac{\psi_{-1}'}{2}\right) = -\frac{C+DY_{-1}}{A-BY_{-1}}\\ -Y_1' &= \tan\left(\frac{\psi_1'}{2}\right) = -\frac{C+DY_1}{A-BY_1}\\ -Y_2' &= \tan\left(\frac{\psi_2'}{2}\right) = -\frac{C+DY_2}{A-BY_2}\end{aligned} \tag{2}$$

where A, jB, jC, and D are the entries of the transfer matrix corresponding to the matching network. If the circulation conditions (1) are used in conjunction with (2), conditions on the Y's or Y''s are obtained. For instance

$$\tan\left(\frac{\psi_1'}{2}\right) = \tan\left(\frac{\psi_0'}{2} + \frac{\pi}{4}\right) = \frac{\tan(\psi_0'/2) + \tan(\pi/4)}{1 - \tan(\psi_0'/2)\tan(\pi/4)}$$

or

$$-Y_1' = \frac{-Y_0' + 1}{1 + Y_0'}.$$

The resulting conditions on the Y''s are

$$\begin{aligned}-Y_1' &= \frac{-Y_0'+1}{1+Y_0'}\\ -Y_{-1}' &= \frac{-Y_0'-1}{1-Y_0'}\\ -Y_2' &= 1/Y_0'.\end{aligned} \tag{3}$$

The first two of these equations are similar to those encountered in the three-port circulator case and will lead us to define an equivalent admittance similar to the three-port equivalent admittance. The last equation imposes an additional condition on the eigenadmittances which can't be fulfilled by matching, as we shall see shortly.

The derivation based on the first two equations will be given for completeness, although it could really be written down directly on the basis of the three-port results [5], [6]. Upon substituting the expressions for the Y''s in terms of the Y's from (2) into the first two equations of (3)

$$-\frac{C+DY_1}{A-BY_1}=\frac{-\dfrac{C+DY_0}{A-BY_0}+1}{1+\dfrac{C+DY_0}{A-BY_0}}$$

$$-\frac{C+DY_{-1}}{A-BY_{-1}}=\frac{-\dfrac{C+DY_0}{A-BY_0}-1}{1-\dfrac{C+DY_0}{A-BY_0}}.$$

As before, one can always choose a reference plane such that $Y_0=j\infty(\psi_0=\pi)$. All starred quantities refer to this choice of the reference plane. In this case, the above two equations reduce to

$$-BC+CD+D^2Y_1^*-BDY_1^*=AD+AB-BDY_1^*-B^2Y_1^*$$

$$-BC-CD-D^2Y_{-1}^*-BDY_{-1}^*=AD-AB-BDY_{-1}^*+B^2Y_{-1}^*$$

or

$$-(AB-CD)=(AD+BC)-(B^2+D^2)Y_1^*$$

$$AB-CD=(AD+BC)+(B^2+D^2)Y_{-1}^*.$$

Adding and subtracting these two equations and setting $AD+BC=1$

$$(B^2+D^2)\left(\frac{Y_1^*-Y_{-1}^*}{2}\right)=1$$

$$AB-CD=(B^2+D^2)\left(\frac{Y_1^*+Y_{-1}^*}{2}\right).$$

But these conditions are the same as those for matching the complex admittance Y_{eq}^* if one makes the identifications

$$\text{Re } Y_{eq}^*=G^*=\frac{Y_1^*-Y_{-1}^*}{2}$$

$$\text{Im } Y_{eq}^*=Y^*=\frac{Y_1^*+Y_{-1}^*}{2}. \tag{4}$$

The third equation of (3) becomes

$$-\frac{C+DY_2^*}{A-BY_2^*}=-\frac{B}{D}$$

or

$$AB-CD=(B^2+D^2)Y_2^*.$$

However, this equation is of the same form as that for Im Y_{eq}^* so that an equation or condition independent of the matching network is obtained.

$$Y_2^*=\frac{Y_1^*+Y_{-1}^*}{2}=Y^*. \tag{5}$$

This condition, which in the above form applies only at the reference plane where $Y_0=j\infty$, must be satisfied before perfect circulation can be obtained by matching. If it is satisfied, then an equivalent admittance can be defined.

III. The Eigenadmittance Condition at an Arbitrary Reference Plane

The eigenadmittance condition (5) and the expressions for the equivalent admittance (4) can readily be generalized to an arbitrary reference plane. This has been done for the equivalent admittance in previous references and will not be repeated here [4], [5]. The generalization of (5) is obtained by expressing

$$Y_i^*=\tan\left(-\frac{\psi_i-\psi_0+\pi}{2}\right), \qquad i=0,\pm 1,2$$

expanding the tangent function using trigonometric formulas, and using (2) to express the results in terms of the eigensusceptances at an arbitrary reference plane Y_0, Y_{-1}, Y_1, and Y_2. The result is

$$\frac{Y_2+1/Y_0}{1-Y_2/Y_0}=\frac{1}{2}\left\{\frac{Y_1+1/Y_0}{1-Y_1/Y_0}+\frac{Y_{-1}+1/Y_0}{1-Y_{-1}/Y_0}\right\}. \tag{6}$$

Upon removing denominators and cancelling common terms,

$$\left(Y_2-\frac{Y_1+Y_{-1}}{2}\right)+\left\{Y_{-1}Y_1+\frac{Y_1Y_2+Y_{-1}Y_2}{2}\right\}\frac{1}{Y_0}+\left\{Y_2-\frac{Y_1+Y_{-1}}{2}\right\}\frac{1}{Y_0^2}+\left\{Y_{-1}Y_1+\frac{Y_2Y_{-1}+Y_1Y_2}{2}\right\}\frac{1}{Y_0^3}=0. \tag{7}$$

This equation is of the form

$$A\left(1+\frac{1}{Y_0^2}\right)+\frac{B}{Y_0}\left(1+\frac{1}{Y_0^2}\right)=0$$

$$\therefore \quad Y_0=-B/A=\frac{Y_1Y_{-1}+\dfrac{Y_2Y_{-1}+Y_1Y_2}{2}}{-Y_2+\dfrac{Y_1+Y_{-1}}{2}}. \tag{8}$$

A more symmetrical form of (8) is

$$(Y_0+Y_2)(Y_1+Y_{-1})=2(Y_1Y_{-1}+Y_0Y_2). \tag{9}$$

Note that an equation of the same form is applicable to eigenimpedances.

If the general admittance condition (9) is satisfied, then it is possible to obtain a perfect four-port circulator by matching. This situation is not too surprising since it is well known that a matched four-port symmetrical device is not necessarily a circulator [7]. Furthermore, for a given magnetic field strength one would expect (9) to be satisfied only at isolated frequencies. This suggests that four-port junction circulators will be narrow band devices in agreement with the experimental situation. One could design narrow band four-port circulators using (9) by determining the eigenadmittances of a four-port junction experimentally using a computerized measurement system as was done in [5] for three-port circulators, and then determining the circumstances under which condition (9) is satisfied. In the case of stripline circulators, one could do the same thing theoretically by using theoretical expressions for the eigenadmittances [8]. In view of the simple application of this condition to stripline hybrids given in the next section, this will not be done here.

Actually, (9) is a sufficient condition but not a necessary condition. Equation (1) assumes that $\psi_2'=\psi_0'+\pi$. Perfect circulation can also be obtained if $\psi_1'=\psi_0'+\pi$ or if $\psi_{-1}'=\psi_0'+\pi$. Consequently, two alternate conditions are obtained if Y_2 is interchanged with Y_1 or Y_{-1} in (9). These conditions are

$$(Y_0+Y_{-1})(Y_1+Y_2)=2(Y_1Y_2+Y_0Y_{-1}) \qquad (10)$$

$$(Y_0+Y_1)(Y_{-1}+Y_2)=2(Y_{-1}Y_2+Y_0Y_1). \qquad (11)$$

IV. An Application to Symmetrical Four-Port Hybrids

One of the nice features of the eigenadmittance approach is that it is applicable to some other devices with symmetry. In particular, it is applicable to certain reciprocal devices which are required to be matched using identical two-port matching networks connected at each port. Consider in particular the 2-branch coupler of Fig. 1. This is a four-port reciprocal device with 2-fold symmetry about two perpendicular axes. It can be shown that the scattering matrix entries S_{11}, S_{12}, S_{13}, and S_{14} are related to the eigenreflection coefficients S_1, S_2, S_3, and S_4 by [9]

$$S_{11}=\tfrac{1}{4}(S_1+S_2+S_3+S_4)$$
$$S_{12}=\tfrac{1}{4}(S_1-S_2+S_3-S_4)$$
$$S_{13}=\tfrac{1}{4}(S_1+S_2-S_3-S_4)$$
$$S_{14}=\tfrac{1}{4}(S_1-S_2-S_3+S_4).$$

If S_1, S_2, S_3, and S_4 are separated by 90° on the unit circle, then a device with this symmetry will be a 90° hybrid. This eigenvalue separation is exactly the same as for a four-port circulator. The results derived for the four-port circulator can be applied directly to the symmetrical four-port hybrid, provided one makes the substitutions $jY_0\to jY_1$, $jY_2\to jY_4$, $jY_1\to jY_2$, and $jY_{-1}\to jY_3$ for the eigenadmittances. The admittance condition corresponding to (9) becomes

$$(Y_1+Y_4)(Y_2+Y_3)=2(Y_2Y_3+Y_1Y_4). \qquad (12)$$

Note that it has been implicitly assumed that port 4 will be the isolated port. If port 2 or port 3 had been taken as the isolated port, then the condition corresponding to (10) or (11) would have had to have been taken. The three different conditions have a more direct interpretation for the hybrid than they do in the case of a four-port circulator.

One of the simplest symmetrical four-port devices is one with the four input ports connected by equal lengths of transmission line of electrical length θ. In Fig. 1 the characteristic admittance of the lines connecting ports 1 and 4 and ports 2 and 3 is Y_0, while the characteristic admittance of the lines connecting ports 1 and 2 and ports 4 and 3 is $Y\cdot Y_0$. It will now be shown that the condition (12) is identically satisfied independent of frequency if $Y=\sqrt{2}$. Expressions for the four eigensusceptances Y_1, Y_2, Y_3, and Y_4 can be found by assuming electric and magnetic walls along the symmetry planes in all combinations. The eigensusceptances become

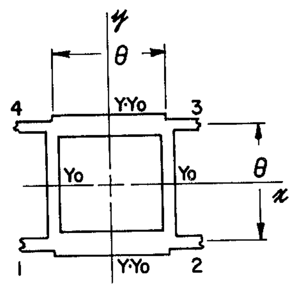

Fig. 1. A diagram of a 2-branch coupler with 2-fold symmetry about the x and y axis. If the ratio Y of the in-line to branch-line admittances is $\sqrt{2}$, then the device can be converted into a 90° hybrid at any frequency by matching the junction at that frequency with identical two-port matching networks connected at each port.

$$Y_1=(Y_0+Y_0Y)\tan\left(\frac{\theta}{2}\right)=Y_0(1+Y)t$$
$$Y_2=-Y_0(1+Y)/t$$
$$Y_3=-Y_0/t+YY_0t=Y_0(Yt-1/t)$$
$$Y_4=Y_0t-YY_0/t=Y_0(t-Y/t).$$

Substituting the above expressions for Y_1, Y_2, Y_3, and Y_4 into (12), one finds that

$$\left(Y\left(t-\frac{1}{t}\right)+2t\right)\left(Y\left(t-\frac{1}{t}\right)-\frac{2}{t}\right)=+2(1+Y)t\left(t-\frac{Y}{t}\right)-2\left(\frac{1+Y}{t}\right)\left(Yt-\frac{1}{t}\right)$$

or

$$\{Y^2+2Y\}\left(t-\frac{1}{t}\right)^2-4=\{2+2Y\}\left(t-\frac{1}{t}\right)^2-4Y^2-4Y+4Y+4$$

$$\therefore\quad Y^2\left(t-\frac{1}{t}\right)^2+4Y^2=2\left(t-\frac{1}{t}\right)^2+8.$$

Notice now the quite surprising result that this equation is satisfied identically independent of t and, hence, of frequency if $Y=\sqrt{2}$. As long as the ratio of the admittance levels of the two different branch sections is $\sqrt{2}$, the coupler can be made to give equal power division by connecting identical two-port matching networks at each port. In a companion paper, a more general result for the case of arbitrary coupling is derived for the structure of Fig. 1.

The equivalent admittance of this junction could be found numerically by evaluating the generalization of (4) to an arbitrary reference plane. Instead, a closed form algebraic expression for it will be taken over directly from a companion paper on directional couplers [10]. This expression is

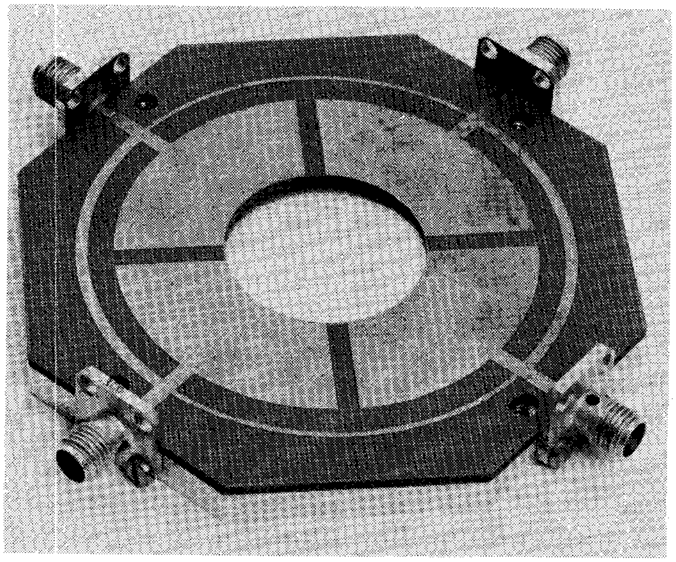

Fig. 2. A picture of a miniaturized 90° stripline hybrid which operates at 400 MHz with line lengths which are a quarter wavelength at 1 GHz. Metallized sectors at each port act as capacitors to match out the junction susceptance at 400 MHz.

$$Y_{eq} = \frac{Y_0\sqrt{Y^2-1}}{\sin\Theta} - jY_0(1+Y)\cot\Theta. \quad (13)$$

For equal power division, $Y=\sqrt{2}$. If we set $Y_0 = G$, then

$$Y_{eq} = \frac{G}{\sin\Theta} - j(1+\sqrt{2})G\cot\Theta. \quad (14)$$

Notice that if the line lengths are a quarter-wavelength long ($\Theta = \pi/2$) and if $Y_0 = G = 1$, then $Y_{eq} = 1$. This must be so since the resulting device is a standard 90° hybrid.

V. A Miniature Stripline Hybrid

The results derived in the previous section lend themselves to a simple and potentially useful application. It has been shown that it is only necessary to match the 2-branch junction of Fig. 1 at any frequency in order to obtain a 90° hybrid at that frequency provided that $Y=\sqrt{2}$. In particular, if the junction can be matched at a low frequency for which the line lengths are much less than a quarter-wavelength, then a quite compact hybrid could be obtained. This can be done easily in two steps. First, the real part of Y_{eq} in (14) must be made equal to one. This will be the case if $G = Y_0 = \sin\Theta$. In stripline, this will require reducing the width of the connecting strips until the junction conductance is one at the selected frequency. Secondly, a capacitative susceptance must be included at each junction to balance out the inductive susceptance $-j(1+\sqrt{2})G\cot\Theta$ of (14). Lumped element capacitors could be used or, in the case of stripline, simply metallized sectors extending in toward the center of the junction.

A picture of such a miniature hybrid constructed in stripline is given in Fig. 2. It was designed to operate at 400 MHz using line lengths which are a quarter-wavelength long at 1 GHz. This corresponds to a linear size reduction of 2.5 or a reduction in surface area of 6.25. The four sectors extending in toward the junction center are the capacitors required to match out the junction susceptance. The line widths have been narrowed to make the conductance 1 at 400 MHz. The device makes efficient use of the available area. In Fig. 3 the experimental return loss versus frequency is given, and in Fig. 4 the experimental coupling to the in-line and branch-line ports is given versus frequency. The bandwidth is about 10 percent for 18-dB return loss and 0.2-dB coupling unbalance. These figures are close to the same as for a standard junction. Reducing the size does not significantly effect the bandwidth. In practice, there will be a practical limit on the size reduction possible to three or so in the linear dimension. For larger reductions the value of Y_0, and hence the width of the connecting strips, becomes impractically small.

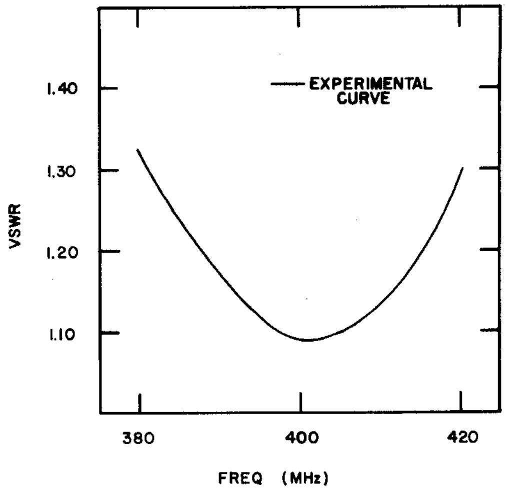

Fig. 3. Experimental return loss of the 400-MHz hybrid versus frequency.

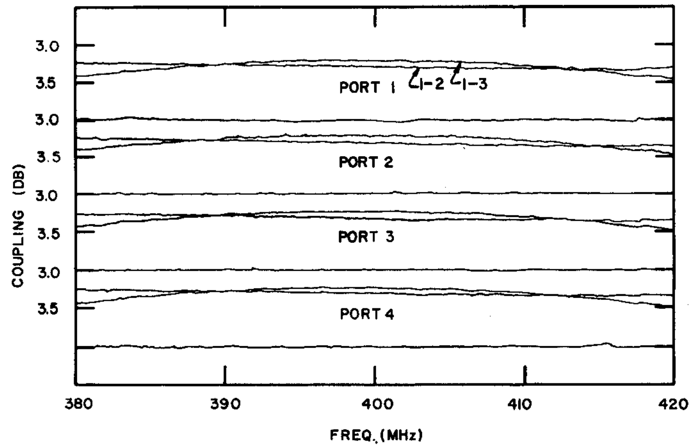

Fig. 4. Experimental coupling to in-line and branch-line ports of the 400-MHz hybrid versus frequency. The level coupling refers to the in-line output port and the parabolic coupling to the branch-line output port with ports 1, 2, 3, and 4 taken as the input port, respectively. The miniaturization procedure does not significantly alter the bandwidth.

VI. Conclusions

A simple condition on the four eigenadmittances of a 4-fold symmetrical nonreciprocal device has been derived which must be fulfilled if perfect circulation is to be obtainable by connecting identical two-port matching networks in each arm. This condition also applies to a reciprocal four-port device with symmetry about two perpendicular axes which is required to be a 90° hybrid upon matching. This is because the eigenreflection coefficients of both devices are separated by 90° on the unit circle. Furthermore, it was shown that this condition is identically satisfied for all frequencies for a 2-branch coupler, provided that the ratio of the in-line to branch-line admittance levels is $\sqrt{2}$. This fact was used to construct a compact stripline hybrid operating at 400 MHz using lines which are a quarter wavelength long at 1 GHz.

VII. Appendix

It is obvious that the results in the text can readily be generalized to the case of the n-port circulator where the n eigenvalue phase angles must be separated by $360/n$ degrees on the unit circle for perfect circulation. Here the results will only be stated for the five-port circulator at the reference plane where $\psi_0 = 180°$. The eigenadmittances at such a plane are $jY_0^* = j\infty, jY_{-1}^*, jY_1^*, jY_{-2}^*$, and jY_2^*. The real and imaginary parts of the equivalent admittance are

$$G^* = \frac{Y_1^* - Y_{-1}^*}{2} \cdot \tan 36° = \frac{Y_2^* - Y_{-2}^*}{2} \cdot \tan 72°$$

$$Y^* = \frac{Y_1^* + Y_{-1}^*}{2} = \frac{Y_2^* + Y_{-2}^*}{2}.$$

It is apparent that two conditions on the eigenadmittances must be satisfied before a circulator can be obtained by matching alone. In the general n-port case, $n-3$ conditions must be satisfied.

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Field Theory Treatment of *H*-Plane Waveguide Junction with Triangular Ferrite Post

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***Abstract*—This paper presents an exact field theory treatment for the *H*-plane waveguide junction with three-sided ferrite prism. The treatment is general, being independent of the geometrical symmetry of the junction, the number of ports, and the location of the ferrite post inside the junction. The solution of the wave equations in the ferrite post and in the surrounding region is written in the form of an infinite summation of cylindrical modes. The fields at the ferrite-air interface are matched using the point-matching technique. This results in two amplitudes for the cylindrical modes describing the fields in the air region in the form of a matrix.**

The fields at the arbitrary boundary between the air region and the waveguides are also matched using the point-matching technique. This results in a finite system of nonhomogeneous equations in the field amplitudes.

The three-port waveguide junction circulator with central triangular ferrite post is analyzed using this technique.

Two specific arrangements are considered. In the first arrangement, the points of the triangles are in the centers of the waveguides, and in the second, the sides of the triangles are in the centers of the waveguides. The method used in this paper can also be applied to study the effect of the ferrite-post geometry on the circulator performance in order to seek the best possible circulator structure.

Excellent agreement has been found between published experimental measurements and the numerical results obtained by this technique in the case of a waveguide junction circulator with cylindrical ferrite post.

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I. Introduction

The VERSATILITY of the most widely used ferrite junction circulator is indicated by the fact that in addition to its use as a circulator, it can also be used as an isolator or as a switch. It can be constructed in either a rectangular waveguide or stripline technique. The waveguide version usually uses a *H*-plane junction.

The problem of the waveguide circulator design has been overcome by numerical techniques for some simpler structures such as latching circulators, simple ferrite-rod *Y* and *T* junctions, and various inhomogeneous ferrite cylinders (ferrite post [8]), ferrite tube–dielectric rod–dielectric sleeve, and ferrite post–metal pin–dielectric sleeve [1]–[3]).

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